JOURNAL OF THE
CHUNGCHEONG MATHEMATICAL SOCIETY
Volume **29**, No. 4, November 2016
http://dx.doi.org/10.14403/jcms.2016.29.4.531

THE JORDAN DERIVATIONS OF SEMIPRIME RINGS AND NONCOMMUTATIVE BANACH ALGEBRAS

Byung-Do Kim*

Abstract. Let R be a 3!-torsion free noncommutative semiprime ring, and suppose there exists a Jordan derivation $D : R \to R$ such that $[[D(x), x], x]D(x) = 0$ or $D(x)[[D(x), x], x] = 0$ for all $x \in R$. In this case we have $[D(x), x]^3 = 0$ for all $x \in R$. Let A be a noncommutative Banach algebra. Suppose there exists a continuous linear Jordan derivation $D : A \to A$ such that $[[D(x), x], x]D(x) \in \mathrm{rad}(A)$ or $D(x)[[D(x), x], x] \in \mathrm{rad}(A)$ for all $x \in A$. In this case, we show that $D(A) \subseteq \mathrm{rad}(A)$.

1. Introduction

Throughout, R represents an associative ring and A will be a real or complex Banach algebra. We write $[x, y]$ for the commutator $xy - yx$ for x, y in a ring. Let $\mathrm{rad}(R)$ denote the (*Jacobson*) *radical* of a ring R. And a ring R is said to be (*Jacobson*) *semisimple* if its Jacobson radical $\mathrm{rad}(R)$ is zero.

A ring R is called *n-torsion free* if $nx = 0$ implies $x = 0$. Recall that R is *prime* if $aRb = (0)$ implies that either $a = 0$ or $b = 0$, and is *semiprime* if $aRa = (0)$ implies $a = 0$. On the other hand, let X be an element of a normed algebra. Then for every $x \in X$ the *spectral radius* of x, denoted by $r(x)$, is defined by $r(x) = \inf\{||x^n||^{\frac{1}{n}} : n \in \mathbb{N}\}$. It is well-known that the following theorem holds: if x be an element of a normed algebra, then $r(x) = \lim_{n\to\infty} ||x^n||^{\frac{1}{n}}$ (see Bonsall and Duncan[1]).

An additive mapping D from R to R is called a *derivation* if $D(xy) = D(x)y + xD(y)$ holds for all $x, y \in R$. And an additive mapping D from R to R is called a *Jordan derivation* if $D(x^2) = D(x)x + xD(x)$ holds for all $x \in R$.

Received March 10, 2016; Accepted October 17, 2016.
2010 Mathematics Subject Classification: Primary 16N60, 16W25, 17B40.
Key words and phrases: Banach algebra, Jordan derivation, prime and semiprime ring, (Jacobson) radical.

Johnson and Sinclair[5] proved that any linear derivation on a semisimple Banach algebra is continuous. A result of Singer and Wermer[9] states that every continuous linear derivation on a commutative Banach algebra maps the algebra into its radical. From these two results, we can conclude that there are no nonzero linear derivations on a commutative semisimple Banach algebra.
Thomas[10] proved that any linear derivation on a commutative Banach algebra maps the algebra into its radical.

A noncommutative version of Singer and Wermer's Conjecture states that every continuous linear derivation on a noncommutative Banach algebra maps the algebra into its radical.

Vukman[11] proved the following: let R be a 2-torsion free prime ring. If $D : R \longrightarrow R$ is a derivation such that $[D(x), x]D(x) = 0$ for all $x \in R$, then $D = 0$.

Moreover, using the above result, he proved that the following holds: let A be a noncommutative semisimple Banach algebra. Suppose that $[D(x), x]D(x) = 0$ holds for all $x \in A$. In this case, $D = 0$.

Kim[6] showed that the following result holds: let R be a 3!-torsion free semiprime ring. Suppose there exists a Jordan derivation $D : R \to R$ such that

$$[D(x), x]D(x)[D(x), x] = 0$$

for all $x \in R$. In this case, we have $[D(x), x]^5 = 0$ for all $x \in R$.

And, Kim[7] has showed that the following result holds:let A be a noncommutative Banach algebra. Suppose there exists a continuous linear Jordan derivation $D : A \to A$ such that $D(x)[D(x), x]D(x) \in \mathrm{rad}(A)$ for all $x \in A$. In this case, we have $D(A) \subseteq \mathrm{rad}(A)$.

In this paper, our aim is to prove the following results in the ring theory in order to apply it to the Banach algebra theory:

let R be a 3!-torsion free semiprime ring.

Suppose there exists a Jordan derivation $D : R \longrightarrow R$ such that

$$[[D(x), x], x]D(x) = 0 \text{ or } D(x)[[D(x), x], x] = 0$$

for all $x \in R$. In this case, we obtain $[D(x), x]^3 = 0$ for all $x \in R$.

Let A be a noncommutative Banach Algebra. Suppose there exists a continuous linear Jordan derivation $D : A \longrightarrow A$ such that

$$[[D(x), x], x]D(x) \in \text{ or } D(x)[[D(x), x], x] \in \mathrm{rad}(A)$$

for all $x \in A$. In this case, we obtain $D(A) \subseteq \mathrm{rad}(A)$ for all $x \in A$.

2. Preliminaries

In this section, we review the basic results in semiprime rings.
The following lemma and theorem are due to Chung and Luh[4].

LEMMA 2.1. *Let R be a $n!$-torsion free ring. Suppose there exist elements $y_1, y_2, \cdots, y_{n-1}, y_n$ in R such that $\sum_{k=1}^{n} t^k y_k = 0$ for all $t = 1, 2, \cdots, n$. Then we have $y_k = 0$ for every positive integer k with $1 \leq k \leq n$.*

THEOREM 2.2. *Let R be a semiprime ring with a derivation D. Suppose there exists a positive integer n such that $(Dx)^n = 0$ for all $x \in R$ and suppose R is $(n-1)!$-torsion free. Then $D = 0$.*

And in 1988, the following statement was obtained by Brešar[3].

THEOREM 2.3. *Let R be a 2-torsion free semiprime ring and let $D : R \longrightarrow R$ be a Jordan derivation. In this case, D is a derivation.*

We denote by $Q(A)$ the set of all quasinilpotent elements in a Banach algebra.
Bresar[2] also proved the following theorem.

THEOREM 2.4. *Let D be a bounded derivation of a Banach algebra A. Suppose that $[D(x), x] \in Q(A)$ for every $x \in A$. Then D maps A into rad(A).*

3. Main results

We need the following notations. After this, by S_m we denote the set $\{k \in \mathbb{N} \mid 1 \leq k \leq m\}$ where m is a positive integer. when R is a ring, we shall denote the maps $B : R \times R \longrightarrow R,\ f, g : R \longrightarrow R$ by $B(x, y) = [D(x), y] + [D(y), x], f(x) = [D(x), x],\ g(x) = [f(x), x], h(x) = [g(x), x] = [[[f(x), x], x] = [[[D(x), x], x], x]$ for all $x, y \in R$ respectively. And we have the basic properties:

$$[yD(x), x] = yf(x) + [y, x]D(x), [D(x)y, x] = f(x)y + D(x)[y, x],$$
$$[[yD(x), x], x] = [yf(x) + [y, x]D(x), x] = [yf(x), x] + [[y, x]D(x), x]$$
$$= yg(x) + 2[y, x]f(x) + [[y, x], x]D(x),$$

$$[[D(x)y,x],x] = [f(x)y + D(x)[y,x],x] = [f(x)y,x] + [D(x)[y,x],x]$$
$$= g(x)y + 2f(x)[y,x] + D(x)[[y,x],x],$$
$$B(x,y) = B(y,x),$$
$$B(x,yz) = B(x,y)z + yB(x,z) + D(y)[z,x] + [y,x]D(z),$$
$$B(x,x) = 2f(x),\ B(x,x^2) = 2(f(x)x + xf(x)),$$
$$[B(x,x^2),x] + [f(x),x^2] = 3(g(x)x + xg(x)),\ x,y,z \in R.$$

$$B(x,yx) = B(x,y)x + 2yf(x) + [y,x]D(x),$$
$$B(x,xy) = xB(x,y) + 2f(x)y + D(x)[y,x],$$
$$B(x,yD(x)) = B(x,y)D(x) + yF(x) + D(y)f(x) + [y,x]D^2(x),$$
$$B(x,D(x)y) = D(x)B(x,y) + F(x)y + f(x)D(y) + D^2(x)[y,x],$$
$$x,y \in R.$$

THEOREM 3.1. *Let R be a 3!-torsionfree noncommutative semiprime ring. Suppose there exists a Jordan derivation $D : R \longrightarrow R$ such that*

$$[[D(x),x],x]D(x) = 0$$

for all $x \in R$. Then we have $D(x) = 0$ for all $x \in R$.

Proof. By Theorem 2.3, we can see that D is a derivation on R. From the assumption,

$$[[D(x),x],x]D(x) = g(x)D(x) = [f(x),x]D(x) = 0,\ x \in R. \tag{3.1}$$

Replacing $x + ty$ for x in (3.1), we have

$$\begin{aligned} &[[D(x+ty),x+ty]D(x+ty) \\ &\equiv [[D(x),x],x]D(x) + t\{[B(x,y),x]D(x) \\ &+[f(x),y]D(x) + g(x)D(y)\} + t^2 J_1(x,y) \\ &+t^3 J_2(x,y) + t^4 g(y)D(y) \\ &= 0,\ x,y \in R, t \in S_3 \end{aligned} \tag{3.2}$$

where $J_i, 1 \le i \le 3$, denotes the term satisfying the identity (3.2). From (3.1) and (3.2), we obtain

$$\begin{aligned} &t\{[B(x,y),x]D(x) + [f(x),y]D(x) + g(x)D(y)\} \\ &+t^2 J_1(x,y) + t^3 J_2(x,y) \\ &= 0,\ x,y \in R, t \in S_3. \end{aligned} \tag{3.3}$$

Since R is 3!-torsionfree, by Lemma 2.1 the relation (3.3) yields

$$\begin{aligned} &[B(x,y),x]D(x) + [f(x),y]D(x) + g(x)D(y) \\ &= 0, x,y \in R. \end{aligned} \tag{3.4}$$

Let $y = x^2$ in (3.4). Then using (1), (3.1), we get

$$\begin{aligned}(3.5)\qquad & 2\{[f(x)x + xf(x), x]\}D(x) + (g(x)x + xg(x))D(x)\\ & +g(x)(D(x)x + xD(x))\\ & = 2g(x)xD(x) + 2xg(x)D(x) + gxD(x) + xg(x)D(x)\\ & +gD(x)x + g(x)xD(x)\\ & = 4g(x)xD(x) + 3xg(x)D(x) +\\ & +gD(x)x\\ & = 4g(x)xD(x) = 4h(x)D(x) = -4g(x)f(x)\\ & = 0, x \in R.\end{aligned}$$

Since R is 3!-torsion free, it follows from (3.5) that

$$(3.6)\qquad h(x)D(x) = g(x)f(x) = 0, x \in R.$$

Substituting xy for y in (3.4), we arrive at

$$\begin{aligned}(3.7)\qquad & [xB(x, y) + 2f(x)y + D(x)[y, x], x]D(x) + g(x)yD(x)\\ & +x[f(x), y]D(x) + g(x)xD(y) + g(x)D(x)y\\ & = x[B(x, y), x]D(x) + 2f(x)[y, x]D(x) + 2g(x)yD(x)\\ & +D(x)[[y, x], x]D(x) + f(x)[y, x]D(x)\\ & +g(x)yD(x) + x[f(x), y]D(x) + g(x)xD(y) + g(x)D(x)y\\ & = x[B(x, y), x]D(x) + 3f(x)[y, x]D(x) + 3g(x)yD(x)\\ & +D(x)[[y, x], x]D(x) + x[f(x), y]D(x) + g(x)xD(y)\\ & +g(x)D(x)y = 0, x, y \in R.\end{aligned}$$

Left multiplication of (3.4) by x leads to

$$\begin{aligned}(3.8)\qquad & x[B(x, y), x]D(x) + x[f(x), y]D(x) + xg(x)D(y)\\ & = 0, x, y \in R.\end{aligned}$$

Combining (3.1), (3.7) with (3.8),

$$\begin{aligned}(3.9)\qquad & 3f(x)[y, x]D(x) + 3g(x)yD(x) + D(x)[[y, x], x]D(x)\\ & +h(x)D(y) = 0, x, y \in R.\end{aligned}$$

Writing $yD(x)$ for y in (3.9), we have

$$\begin{aligned}(3.10)\qquad & 3f(x)[y, x]D(x)^2 + 3f(x)yf(x)D(x) + 3g(x)yD(x)^2\\ & +D(x)yg(x)D(x) + 2D(x)[y, x]f(x)D(x)\\ & +D(x)[[y, x], x]D(x)^2 + h(x)D(y)D(x)\\ & +h(x)yD^2(x) = 0, x, y \in R.\end{aligned}$$

Right multiplication of (3.9) by $D(x)$ gives

$$3f(x)[y,x]D(x)^2 + 3g(x)yD(x)^2 + D(x)[[y,x],x]D(x)^2 + h(x)D(y)D(x) = 0, x, y \in R. \tag{3.11}$$

From (3.10) and (3.11), we get

$$3f(x)yf(x)D(x) + D(x)yg(x)D(x) + 2D(x)[y,x]f(x)D(x) + h(x)yD^2(x) = 0, x, y \in R. \tag{3.12}$$

From (3.1) and (3.12), one obtains

$$3f(x)yf(x)D(x) + 2D(x)[y,x]f(x)D(x) + h(x)yD^2(x) = 0, x, y \in R. \tag{3.13}$$

Right multiplication of (3.9) by x yields

$$3f(x)[y,x]D(x)x + 3g(x)yD(x)x + D(x)[[y,x],x]D(x)x + h(x)D(y)x = 0, x, y \in R. \tag{3.14}$$

Putting yx instead of y in (3.9), we have

$$3f(x)[y,x]xD(x) + 3g(x)yxD(x) + D(x)[[y,x],x]xD(x) + h(x)D(y)x + h(x)yD(x) = 0, x, y \in R. \tag{3.15}$$

From (3.14) and (3.15),

$$3f(x)[y,x]f(x) + 3g(x)yf(x) + D(x)[[y,x],x]f(x) - h(x)yD(x) = 0, x, y \in R. \tag{3.16}$$

Let $y = D(x)$ in (3.16). Then we obtain

$$3f(x)^3 + 3g(x)D(x)f(x) + D(x)g(x)f(x) + h(x)D(x)^2 = 0, x, y \in R. \tag{3.17}$$

From (3.1),(3.6) and (3.17), we get

$$3f(x)^3 = 0, x \in R. \tag{3.18}$$

Since R is 3!-torsionfree, (3.18) yields

$$f(x)^3 = 0, x \in R. \tag{3.19}$$

□

THEOREM 3.2. *Let R be a 3!-torsionfree noncommutative semiprime ring. Suppose there exists a Jordan derivation $D : R \longrightarrow R$ such that*

$$D(x)[[D(x),x],x] = 0$$

for all $x \in R$. Then we have $D(x) = 0$ for all $x \in R$.

Proof. By Theorem 2.3, we can see that D is a derivation on R. From the assumption,

$$D(x)[[D(x),x],x] = D(x)g(x) = D(x)[f(x),x] = 0,\ x \in R. \tag{3.20}$$

Replacing $x+ty$ for x in (3.20), we have

$$\begin{aligned} &D(x+ty)[[D(x+ty),x+ty] \\ &\equiv D(x)[[D(x),x],x] + t\{D(y)g(x) + D(x)[B(x,y),x] \\ &+D(x)[f(x),y]\} + t^2K_1(x,y) \\ &+t^3K_2(x,y) + t^4D(y)g(y) \\ &= 0,\ x,y \in R, t \in S_3 \end{aligned} \tag{3.21}$$

where $K_i, 1 \le i \le 3$, denotes the term satisfying the identity (3.21). From (3.20) and (3.21), we obtain

$$\begin{aligned} &t\{D(y)g(x) + D(x)[B(x,y),x] + D(x)[f(x),y]\} \\ &+t^2K_1(x,y) + t^3K_2(x,y) \\ &= 0,\ x,y \in R, t \in S_3. \end{aligned} \tag{3.22}$$

Since R is 3!-torsionfree, by Lemma 2.1 the relation (3.22) yields

$$\begin{aligned} &D(y)g(x) + D(x)[B(x,y),x] + D(x)[f(x),y] \\ &= 0, x, y \in R. \end{aligned} \tag{3.23}$$

Let $y = x^2$ in (3.23). Then using (3.20), we get

$$\begin{aligned} &\{D(x)x + xD(x)\}g(x) + 2D(x)\{[f(x)x + xf(x),x]\} \\ &+D(x)(g(x)x + xg(x)) \\ &= D(x)xg(x) + xD(x)g(x) + 2D(x)g(x)x + 2D(x)xg(x) \\ &+D(x)g(x)x + D(x)xg(x) \\ &= 4D(x)xg(x) + 3D(x)g(x)x + xD(x)g(x) \\ &= 4f(x)g(x) = -4D(x)h(x) = 0, x \in R. \end{aligned} \tag{3.24}$$

Since R is 3!-torsion free, we obtain from (3.24)

$$f(x)g(x) = D(x)h(x) = 0, x \in R. \tag{3.25}$$

Right multiplication of (3.23) by x leads to

$$\begin{aligned} &D(y)g(x)x + D(x)[B(x,y),x]x + D(x)[f(x),y]x \\ &= 0, x, y \in R. \end{aligned} \tag{3.26}$$

Substituting yx for y in (3.23), we have

$$\begin{aligned}(3.27)\qquad & D(y)xg(x)+yD(x)g(x)+D(x)[B(x,y)x+2yf(x)\\ &+[y,x]D(x),x]+D(x)[f(x),y]x+D(x)yg(x)\\ &=D(y)xg(x)+yD(x)g(x)+D(x)[B(x,y),x]x\\ &+2D(x)yg(x)+2D(x)[y,x]f(x)+D(x)[y,x]f(x)\\ &+D(x)[[y,x],x]D(x)+D(x)[f(x),y]x+D(x)yg(x)\\ &=D(y)xg(x)+yD(x)g(x)+D(x)[B(x,y),x]x\\ &+3D(x)yg(x)+3D(x)[y,x]f(x)\\ &+D(x)[[y,x],x]D(x)+D(x)[f(x),y]x\\ &=0, x,y\in R.\end{aligned}$$

From (3.20), (3.26) and (3.27), we arrive at

$$\begin{aligned}(3.28)\qquad & D(y)h(x)+3D(x)yg(x)+3D(x)[y,x]f(x)\\ &+D(x)[[y,x],x]D(x)=0, x,y\in R.\end{aligned}$$

Replacing $D(x)y$ for y in (3.28), we obtain

$$\begin{aligned}(3.29)\qquad & D(x)D(y)h(x)+D^2(x)yh(x)+3D(x)^2yg(x)\\ &+3D(x)^2[y,x]f(x)+3D(x)f(x)yf(x)\\ &+D(x)^2[[y,x],x]D(x)+2D(x)f(x)[y,x]D(x)\\ &+D(x)g(x)yD(x)=0, x,y\in R.\end{aligned}$$

Left multiplication of (3.28) by $D(x)$ gives

$$\begin{aligned}(3.30)\qquad & D(x)D(y)h(x)+3D(x)^2yg(x)+3D(x)^2[y,x]f(x)\\ &+D(x)^2[[y,x],x]D(x)=0, x,y\in R.\end{aligned}$$

From (3.29) and (3.30), it follows that

$$\begin{aligned}(3.31)\qquad & D^2(x)yh(x)+3D(x)f(x)yf(x)+2D(x)f(x)[y,x]D(x)\\ &+D(x)g(x)yD(x)=0, x,y\in R.\end{aligned}$$

From (3.20) and (3.31), we get

$$\begin{aligned}(3.32)\qquad & D^2(x)yh(x)+3D(x)f(x)yf(x)+2D(x)f(x)[y,x]D(x)\\ &+D(x)g(x)yD(x)=0, x,y\in R.\end{aligned}$$

Writing xy for y in (3.28), we arrived at

$$\begin{aligned}(3.33)\qquad & xD(y)h(x)+D(x)yh(x)+3D(x)xyg(x)+3D(x)x[y,x]f(x)\\ &+D(x)x[[y,x],x]D(x)=0, x,y\in R.\end{aligned}$$

Left multiplication of (3.28) by x yields

$$xD(y)h(x) + 3xD(x)yg(x) + 3xD(x)[y,x]f(x) + xD(x)[[y,x],x]D(x) = 0, x, y \in R. \tag{3.34}$$

From (3.33) and (3.34), one obtains

$$D(x)yh(x) + 3f(x)yg(x) + 3f(x)[y,x]f(x) + f(x)[[y,x],x]D(x) = 0, x, y \in R. \tag{3.35}$$

Let $y = D(x)$ in (3.35). Then we have

$$D(x)^2h(x) + 3f(x)D(x)g(x) + 3f(x)^3 + f(x)g(x)D(x) = 0, x, y \in R. \tag{3.36}$$

From (3.20),(3.25) and (3.36), we get

$$3f(x)^3 = 0, x \in R. \tag{3.37}$$

Since R is 3!-torsionfree, (3.37) gives

$$f(x)^3 = 0, x \in R.$$

□

Combining Vukman's idea [12] with and Brešar [2] and Kim's idea [6], we have the following theorem from the simple calculations.

THEOREM 3.3. *Let A be a Banach algebra. Suppose there exists a continuous linear Jordan derivation $D : A \longrightarrow A$ such that*

$$[[D(x), x], x]D(x) \in rad(A)$$

for all $x \in A$. Then we have $D(A) \subseteq rad(A)$.

Proof. It suffices to prove the case that A is noncommutative. By the result of B.E. Johnson and A.M. Sinclair [5] any linear derivation on a semisimple Banach algebra is continuous. Sinclair [9] has proved that every continuous linear Jordan derivation on a Banach algebra leaves the primitive ideals of A invariant. Hence for any primitive ideal $P \subseteq A$ one can introduce a derivation $D_P : A/P \longrightarrow A/P$, where A/P is a prime and factor Banach algebra, by $D_P(\hat{x}) = D(x) + P$, $\hat{x} = x + P$. By the assumption that $[[D(x), x], x]D(x) \in \mathrm{rad}(A)$, $x \in A$, we obtain $[[D_P(\hat{x}), \hat{x}], \hat{x}]D_P(\hat{x}) = 0$, $\hat{x} \in A/P$, since all the assumptions of Theorem 3.1 is fulfilled. Let the factor prime Banach algebra A/P be noncommutative. Then from Theorem 3.1 we have $[D_P(\hat{x}), \hat{x}]^3 = 0$, $\hat{x} \in A/P$. Hence by using Theorem 2.4, we get $D_P(\hat{x}) \in \mathrm{rad}(A/P) = \{0\}$, $\hat{x} \in A/P$. Thus we obtain $D(x) \in P$ for all $x \in A$ and all primitive ideals P of A. Hence $D(A) \subseteq \mathrm{rad}(A)$. And we consider the case that A/P

is commutative. Then since A/P is a commutative Banach semisimple Banach algebra, from the result of B.E. Johnson and A.M. Sinclair [5], it follows that $D_P(\hat{x}) = 0,\ \hat{x} \in A/P$. And so, $D(x) \in P$ for all $x \in A$ and all primitive ideals P of A. Hence $D(A) \subseteq \mathrm{rad}(A)$. Therefore in any case we obtain $D(A) \subseteq \mathrm{rad}(A)$. □

The following theorem is similarly proved in the above proof of Theorem 3.3.

THEOREM 3.4. *Let A be a (noncommutative) Banach algebra. Suppose there exists a continuous linear Jordan derivation $D : A \longrightarrow A$ such that*

$$D(x)[[D(x), x], x] \in rad(A)$$

for all $x \in A$. Then we have $D(A) \subseteq rad(A)$.

THEOREM 3.5. *Let A be a semisimple Banach algebra. Suppose there exists a linear Jordan derivation $D : A \longrightarrow A$ such that*

$$[[D(x), x], x]D(x) = 0$$

for all $x \in A$. Then we have $D = 0$.

Proof. It suffices to prove the case that A is noncommutative. According to the result of B.E. Johnson and A.M. Sinclair[5] every linear derivation on a semisimple Banach algebra is continuous. A.M. Sinclair[9] proved that any continuous linear derivation on a Banach algebra leaves the primitive ideals of A invariant. Hence for any primitive ideal $P \subseteq A$ one can introduce a derivation $D_P : A/P \longrightarrow A/P$, where A/P is a prime and factor Banach algebra, by $D_P(\hat{x}) = D(x) + P$, $\hat{x} = x + P$. From the given assumptions $[[D(x), x], x]D(x) = 0,\ x \in A$, it follows that $[[D_P(\hat{x}), \hat{x}], \hat{x}]D_P(\hat{x}) = 0,\ \hat{x} \in A/P$, since all the assumptions of Theorem 3.1 is fulfilled. The factor algebra A/P is noncommutative, by Theorem 3.1 we have $[D_P(\hat{x}), \hat{x}]^3 = 0,\ \hat{x} \in A/P$. Thus we obtain $[D_P(\hat{x}), \hat{x}] \in Q(A/P)$. Then by Theorem 2.4, we obtain $D_P(\hat{x}) \in \mathrm{rad}(A/P) = \{0\}$ for all $\hat{x} \in A/P$ and all primitive ideals P of A. That is, $D(x) \in P$ for all $x \in A$ and primitive ideals P in A. Hence we get $D(A) \subseteq P$ for all primitive ideals P of A. Therefore $D(A) \subseteq \mathrm{rad}(A)$. But since A is semisimple, $D = 0$. □

The following theorem is similarly proved in the above proof of theorem.

THEOREM 3.6. *Let A be a semisimple Banach algebra. Suppose there exists a linear Jordan derivation $D : A \longrightarrow A$ such that*

$$D(x)[[D(x), x], x] = 0$$

for all $x \in A$. *Then we have* $D = 0$.

As a special case of Theorem 3.3 we have the following result which characterizes commutative semisimple Banach algebras.

Corollary 3.7. *Let* A *be a semisimple Banach algebra. Suppose*

$$[[[x, y], x], x][x, y] = 0$$

for all $x, y \in A$. *In this case,* A *is commutative.*

As a special case of Theorem 3.5 we get the following statement which characterizes commutative semisimple Banach algebras.

Corollary 3.8. *Let* A *be a semisimple Banach algebra. Suppose*

$$[x, y][[[x, y], x], x] = 0$$

for all $x, y \in A$. *In this case,* A *is commutative.*

References

[1] F. F. Bonsall and J. Duncan, *Complete Normed Algebras*, Berlin-Heidelberg-New York, 1973.

[2] M. Brešar, *Derivations of noncommutative Banach algebras II*, Arch. Math. **63** (1994), 56-59.

[3] M. Brešar, *Jordan derivations on semiprime rings*, Proc. Amer. Math. Soc. **104** (1988), no. 4, 1003-1006.

[4] L. O. Chung and J. Luh, *Semiprime rings with nilpotent derivatives,* Canad. Math. Bull. **24** (1981), no. 4, 415-421.

[5] B. E. Johnson and A. M. Sinclair, *Continuity of derivations and a problem of Kaplansky*, Amer. J. Math. **90** (1968), 1067-1073.

[6] B. D. Kim, *On the derivations of semiprime rings and noncommutative Banach algebras*, Acta Mathematica Sinica **16** (2000), no. 1, 21-28.

[7] B. D. Kim, *Derivations of semiprime rings and noncommutative Banach algebras*, Commun. Korean Math. Soc **17** (2002), no. 4, 607-618.

[8] B. D. Kim, *Jordan derivations of semiprime rings and noncommutative Banach algebras, II*, J. Korea Soc. Math. Educ. Ser. B. Pure Appl. Math. **15** (2008), no. 3, 259-296.

[9] A. M. Sinclair, *Jordan homohorphisms and derivations on semisimple Banach algebras*, Proc. Amer. Math. Soc. **24** (1970), 209-214.

[10] I. M. Singer and J. Wermer, *Derivations on commutative normed algebras*, Math. Ann. **129** (1955), 260-264.

[11] J. Vukman, *A result concerning derivations in noncommutative Banach algebras*, Glasnik Mathematicki **26** (1991), 83-88.

[12] J. Vukman, *On derivations in prime rings and Banach algebras*, Proc. Amer. Math. Soc. **116** (1992), no. 4, 877-884.

*
Department of Mathematics
Gangneung-Wonju National University
Gangneung 25457, Republic of Korea
E-mail: bdkim@gwnu.ac.kr